

# Math 206A Lecture 8 Notes

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## 1 Linear Algebra Methods and the Kahn-Kalai Theorem

### 1.1 Linear algebra methods

In the proof of the Kahn-Kalai theorem, we have  $M \subseteq \{\pm 1\}^n \subseteq \mathbb{R}^n$  with  $|M| = 2^{n-2}$ . We want the maximal subset  $A$  such that  $a \cdot a' \neq 0$  for all  $a, a' \in A$ . We will show that this is less than  $c^n$ , where  $c < 2$ . We get that the number of parts in the Borsuk part of  $M \otimes M > 2^{n-2}/c^n$ .

**Theorem 1.1** (odd town theorem). *Suppose  $\mathcal{A} = \{A_1, \dots, A_N\} \subseteq \mathcal{P}(\{1, \dots, n\})$  is a collection such that  $|A_i|$  is odd for all  $i$ , and  $|A_i \cap A_j|$  is even for all  $i < j$ . Then  $|\mathcal{A}| \leq n$ .*

*Proof.* Let  $v_i$  be the characteristic vector of  $A_i$  in  $\mathbb{Q}^n$ . For example, if  $A_1 = \{1, 4, 5\}$  and  $n = 5$ , then  $v_1 = (1, 0, 0, 1, 1)^\top$ . Then  $\|v_i\|^2 = 1 \pmod{2}$ , and  $v_i \cdot v_j = 0 \pmod{2}$  if  $i \neq j$ . We claim that the  $v_i$  are linearly independent as vectors in  $F_2^n$ . Assume  $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ . We can take the  $\lambda_i$  to be integers, and without loss of generality,  $\lambda_1$  is odd. Then  $\lambda_1 = \lambda_1 \|v_1\|^2 + \dots + \lambda_n \langle v_n, v_1 \rangle = 0 \pmod{2}$ , which is a contradiction.  $\square$

**Theorem 1.2** (2-distance theorem). *Let  $X \subseteq \mathbb{R}^n$  be such that  $d(x, x') \in \{a, b\}$  for all  $x \neq x'$  and  $x, x' \in X$ . Then  $|X| = O(n^2)$ .*

When the number of possible distances is 1 instead of 2, we get that  $|X| \leq n + 1$ , since  $X$  must be the vertices of a simplex.

*Proof.* Let  $X = \{z_1, \dots, z_N\}$  and  $F(x, y) := (|x - y|^2 - a^2)(|x - y|^2 - b^2)$ . Then

$$F(z_i, z_j) = \begin{cases} a^2 b^2 & i = j \\ 0 & i \neq j. \end{cases}$$

Define  $f_i(y) := F(z_i, y)$ . Then the  $f_i$  are linearly independent. Indeed, suppose  $\lambda_1 f_1 + \dots + \lambda_N f_N = 0$ . Then  $\lambda_1 f_1(z_1) = 0$ , so  $\lambda_1 = 0$ . This is true for all  $i$ . So the number of  $f_i$  is at most the dimension of the space containing the  $f_i$ . So  $N = O(n^2)$ .  $\square$

## 1.2 Kahn-Kalai using linear algebra methods

Let's continue with the proof of the Kahn-Kalai theorem. Let  $M = \{x_1 = 1, x_2, \dots, x_n \in \{\pm 1\}, x_2 \cdots x_n = 1\}$ . We also had  $n = 4p$ , where  $p$  is prime.

**Lemma 1.1.** *Let  $A \subset M$  be such that  $a \cdot a' \neq 0$  for  $a, a' \in A$ . Then  $|A| \leq 2^{n/2}$ .*

*Proof.* Define  $G(t) = (t-1)(t-2)\cdots(t-p+1)$ . Let  $V \subseteq \mathbb{Q}[x_2, \dots, x_n]$  be the subspace of squarefree polynomials with  $\deg \leq n/4 = p$ ; that is, the monomials generating  $V$  have no  $x_i$  to a square or higher power. We will show that  $W \subseteq V \implies \dim(W) \leq 2^{n/4}(n/4)$ . Note that  $\dim V = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n/4} < 2^{n/2}$ .

Let  $F_a = G(a \cdot (1, z_2, \dots, z_n))$  for  $a \in A$ ; this is really a polynomial in  $z_2, \dots, z_n$ . By definition,  $F_a \in V$ . Next time, we will show that the  $F_a$  are linearly independent, which will produce a bound on  $|A|$ .  $\square$